The point of origin of your journey along the line is the **number 0**. All other numbers are measured by how close or far they are from 0. Just a step away, the **number 1** sets the direction and scale of measurement. Keep taking those same steps to reach all the **counting numbers**; or back up as well, and you'll encounter all of the **integers**, the mileposts of the Number Line.

The integers relate to one another in a never-ending variety of **patterns**. The interplay of those patterns gives each number its own unique character and story. **Pick a number to visit. Which patterns is it a part of?** Now step back and get the big perspective — can you see the layout of an entire pattern? Can you guess the next number in the pattern beyond 100?

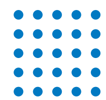
Between the integer mileposts of the Number Line lies the vast countryside of the **Real Numbers.** In this brief guide we've highlighted just a handful of the most interesting attractions. Many more (indeed, infinitely more!) lie waiting to be discovered.

Enjoy your tour!

PATTERNS AMONG THE INTEGER MILEPOSTS

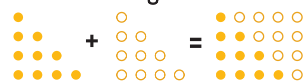
PRIMES Don't miss the prime numbers! Since arithmetic was invented, people have tried to understand how multiplication connects different integers. For example, 15 equals 3 × 5, so 3 and 5 are called "factors" of 15. You can divide most numbers into factors in a similar way. But some numbers, called the "primes," cannot be broken down further. Ancient Greek philosophers named the smallest bits of matter "atoms," which means "indivisible." Primes are the atoms of the number world — they are the indivisible building blocks of all of the integers. As atoms combine to make molecules, so primes can be multiplied together to form all the other integers.

SQUARES The square numbers are another must-see. This fascinating family of numbers comes from multiplying each integer by itself. Imagine, for example, a square made up of five rows of five dots each. That diagram would contain 5 × 5 = 25 dots altogether, so 25 shows up in the pattern of square numbers.



FIBONACCI SEQUENCE In an arithmetic textbook in the year 1202, Leonardo of Pisa (also known as Fibonacci) published an innocuous word problem about raising rabbits and touched off centuries of mathematical discovery. He asked how many pairs of rabbits you would have after a year if you started with one pair, and if every pair at least two months old produced a new pair of rabbits each month. Answering this problem leads to the sequence of numbers shown here, in which each number is the sum of the previous two. Although the sequence doesn't actually tell you much about real rabbits, it does come up over and over again in nature — for example, the number of spirals of seeds in a sunflower is almost always a Fibonacci number.

TRIANGULAR NUMBERS The story is told about the great mathematician Gauss that in 1784 as a schoolboy, his class was assigned to add up every number from 1 to 100. Much to the teacher's amazement, Gauss returned with the correct answer — 5050 — in less than a minute! This sum can be thought of as the number of dots in a triangular array, and two triangles can go together to form a rectangle, easily counted using multiplication. It seems that as a schoolboy, Gauss discovered the formula $n(n+1)/2$ for the nth triangular number. Can you see the pattern in the triangular numbers shown here?

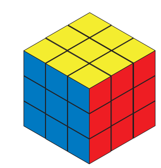


PERFECT NUMBERS At least since the times of the Greek philosopher Pythagoras, visitors to the number line have been comparing numbers to their factors (see the Primes.) One way to compare is to add up all of the smaller factors of a number. For most numbers, like 21, the sum you get (1+3+7 = 11) will be less than the starting number. For many other numbers, like 12, the sum (1+2+3+4+6 = 16) will be more than the original number. And for just a handful of numbers, like 28, the sum comes out exactly equal to the original number: 1+2+4+7+14 = 28. Given their rarity, Greek mathematicians called these numbers "perfect." Every even perfect number must be a triangular number, so we've indicated them with a star on their triangle. Nobody knows whether there are any odd perfect numbers — maybe you can solve that mystery someday!

FACTORIALS The factorials are the multiplication version of the triangular numbers: instead of adding up the first several numbers, you multiply them all together. For example, the 5th factorial number, written 5!, is 1 × 2 × 3 × 4 × 5. All that multiplication makes the factorial numbers get big fast, so you won't see too many on this portion of the Number Line. Factorials come up all the time in calculating the probabilities of things. For example, the chance that you are dealt a spades royal flush in poker is 1 in 52!/5!47!, or 1 in 2,598,960.

POWERS OF TWO You'll definitely want to visit the powers of two! They're what you get when you multiply two by itself again and again, like this: 2 × 2 × 2 × 2 × 2 = 32, which can also be written like this: 2^5 = 32. The powers of two get really big, really fast, because every time the exponent increases by just one, the value doubles! The human population has grown much like this, because each generation multiplies the size of the previous one by some factor. That's what we mean when we say that something is "growing exponentially."

CUBES You've already visited the square numbers. Now extend the idea of that pattern into three dimensions — it will work just as well. For example, 27, which is the number of small blocks in a 3 × 3 × 3 cube like the Rubik's cube below, is one of the cubic numbers. Another way of looking at this is that a cube is anything you can get by multiplying an integer by itself three times. Take a look at the number line. Notice that cubes appear on both sides of 0, but squares all appear to the right of 0. Do you know why?

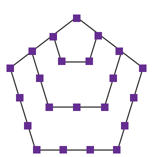


HIGHLY COMPOSITE NUMBERS If primes are the atoms of the number world, then highly composite numbers are at the opposite end of the spectrum — the most complex molecules among numbers. To qualify, a number must have more factors than any smaller number has. You'll recognize some familiar numbers in this category — the number of inches in a foot, the number of seconds in a minute, the number of degrees in a circle. These numbers all show up because they're easy to break down into parts, so they're helpful when you want to work with fractions of a foot or slices of a circle.

PIZZA NUMBERS Now we come to a highly practical and potentially tasty pattern. What's the maximum number of pieces you can slice a pizza into with a specific number of straight-line cuts? The pizza numbers are the answer! Note they start out like the powers of two, but soon you get to a spot where you can't quite cut that many pieces. Next time you have a pizza, see if you can cut it into 11 pieces with just four cuts.

CAKE NUMBERS Cake numbers are just like pizza numbers, only messier. Now you're cutting a cake in three dimensions: you can cut horizontally, vertically, diagonally, or in any other direction as long as you cut straight! As a result, once you're using three cuts or more, the cake numbers are always larger than the pizza numbers. So for example, you can cut a cake into 15 pieces with only four cuts — but some of those pieces would not have much frosting. In fact, one would just be a chunk from the center of the cake!

PENTAGONAL NUMBERS If there are triangular and square numbers, why not pentagonal? In fact, for any regular arrangement of points, you can create a number pattern by counting how many points there are in larger and larger versions of the arrangement.



CONSTRUCTIBLE POLYGON NUMBERS Ancient geometers favored two tools: the compass and the straightedge. They created methods using just these tools for drawing shapes such as equilateral triangles, squares, and pentagons. Millennia later, Gauss (see Triangular Numbers) found a way to draw a regular 17-sided shape, the first polygon construction unknown in classical times. Gauss was so proud of this discovery that he requested the figure be placed on his tombstone. Now we know that only some regular polygons are possible to construct, and this number family tells which.

TETRAHEDRAL NUMBERS The tetrahedral numbers extend the triangular numbers into three dimensions, just as the cubes do for the squares. Look at the difference between each pair of tetrahedral numbers — do you recognize the values you find?

ATTRACTIONS FROM THE COUNTRYSIDE OF REAL NUMBERS

$\sqrt{2}$ = 1.41421356... Some numbers you find in the Real Number countryside are rational — they can be expressed as a fraction with integer numerator and denominator, and their decimal expressions either terminate or enter a repeating pattern. Others, like the square root of 2, are irrational — not equal to any fraction, with decimal expressions that go on forever without repeating. This number is one of the first and simplest irrational numbers encountered: the diagonal of a square with sides of length 1.

ϕ = 1.6180339887... If you start with a rectangle which is not a square, you can cut a square off one end, leaving a smaller rectangle behind. If the rectangle left behind is a scaled-down copy of the original, it is called a Golden Rectangle and the ratio of the lengths of its sides is ϕ, the Golden Ratio.

The number ϕ also shows up as the ratio of the diagonal to the side of a regular pentagon. The Golden Ratio shares deep connections with the Fibonacci sequence. For example, the ratio of each successive Fibonacci number to the previous one gets closer and closer to ϕ.

e = 2.718281828459... Exponential growth (see Powers of Two) can start from any base, but mathematicians use this one special base called e more than any other. Why? The exponential function e^x has the unique property that its rate of growth is given by the identical expression e^x. This property simplifies calculations done with base e, making it the natural choice for modeling exponential growth in nature.

π = 3.14159265... Draw a circle, any circle! Carefully measure the circumference, and divide by the diameter. The ratio you get is always the same, no matter how large or small the circle, or where you draw it. For practical and aesthetic reasons, people have been computing this irrational ratio to ever-greater accuracy since ancient Egyptian times. Modern computing techniques have allowed billions of digits to be determined.

∞ and $-\infty$ What are the outer limits of the Number Line? Where does it end? The true Number Line extends forever in both directions, but alas, our tour must end somewhere. The section shown here is merely a small segment of the whole. So the space beyond the ends is marked with ∞ and $-\infty$, as reminders of the infinite extent of the line that neither the roving monkey nor humans can ever fully explore.

For answers to the questions in this tour guide and more fun number facts and trivia, please visit us online at mathmidway.org

THE MATH MIDWAY GUIDE to the NUMBERLINE

Come visit a world of Platonic perfection!

Like other exotic realms, the Number Line has its own geography, its own special attractions, and its own key sights to see. Some of these highlights are steeped in history, while others are more recent discoveries. The Math Midway Number Line will help you orient yourself in the world of numbers.